February 1963

THE RELATION BETWEEN PITMAN'S ASYMPTOTIC RELATIVE EFFICIENCY OF TWO TESTS AND THE CORRELATION COEFFICIENT BETWEEN THEIR TEST STATISTICS*

Constance van Eeden

Technical Report No. 23

Department of Statistics
and
Division of Biostatistics

University of Minnesota
Minneapolis, Minnesota

*This research was supported in part by the National Science Foundation under Grant Number G-19126 and by the National Institutes of Health under Grant Number 2G-43(C7+8).

THE RELATION BETWEEN PITMAN'S ASYMPTOTIC RELATIVE EFFICIENCY OF TWO TESTS AND THE CORRELATION COEFFICIENT BETWEEN THEIR TEST STATISTICS

by

Constance van Eeden

1. Introduction

It is well known (cf. e.g. Cramér [1], pp. 477-483) that, under certain regularity conditions, the efficiency of an unbiased estimate of a parameter relative to the most efficient unbiased estimate is equal to the square of the correlation coefficient between the two estimates.

In this paper it will be proved that, under certain regularity conditions, the same relation holds between Pitman's asymptotic relative efficiency of a test with respect to a (in a later to be defined sense) best test and the limit of the square of the correlation coefficient between the two test statistics under H_o.

The theorem will be stated and proved in section 2; section 3 contains some examples.

2. The theorem

Let T and T' be two tests for the hypothesis H_o: $\theta=\theta_o$ against the alternative $\theta > \theta_o$. Then the relative efficiency of T' with respect to T is the ratio n/n', where n and n' are the number of observations necessary to give T and T' the same power β for a given level of significance α. The concept of asymptotic relative efficiency is due to Pitman [8]. He considers the limit of n/n' for a sequence of alternatives depending on the sample size and converging to H_o in such a way that the power of both tests converges to a limit < 1. Pitman proved the following theorem (cf. e.g., Noether [6] and [7]).

Let T_n be a test for the hypothesis H_o: $\theta=\theta_o$ against the alternative $\theta > \theta_o$ based on n observations, let t_n be the test statistic and let

$$(2;1) \qquad \begin{cases} \psi_n(\theta) \stackrel{\text{def}}{=} E(t_n|\theta) \\ \sigma_n^2(\theta) \stackrel{\text{def}}{=} \sigma^2(t_n|\theta). \end{cases}$$

Let further θ_n be a sequence of alternatives such that

$$(2;2) \qquad \theta_n = \theta_o + \frac{k}{\sqrt{n}} ,$$

where k is a positive, finite constant independent of n and let the following conditions be satisfied.

A. an ϵ exists such that, for $\theta_o \leqq \theta \leqq \theta_o+\epsilon$, $\psi_n'(\theta)$ exists,

B. $$\lim_{n \to \infty} \frac{\psi_n'(\theta_n)}{\psi_n'(\theta_o)} = 1 ,$$

C. $$\lim_{n \to \infty} \frac{\sigma_n(\theta_n)}{\sigma_n(\theta_o)} = 1 ,$$

D. $$c \stackrel{\text{def}}{=} \lim_{n \to \infty} \frac{\psi_n'(\theta_o)}{\sqrt{n}\,\sigma_n(\theta_o)}$$ exists and is positive ,

E. the distribution of $\frac{t_n-\psi_n(\theta)}{\sigma_n(\theta)}$ tends to the normal distribution with mean 0 and variance 1, uniform in θ.

Condition E can be replaced by

E'. the distribution of $\frac{t_n-\psi_n(\theta_n)}{\sigma_n(\theta_n)}$ tends to the normal distribution with mean 0 and variance 1.

If T_n and T_n' are two tests both satisfying these conditions then Pitman proved that for the asymptotic relative efficiency $e(T', T)$ of T_n' with respect to T_n we have

$$(2;3) \qquad e(T', T) = \left(\frac{c'}{c}\right)^2 .$$

Now let u be a normal random variable with mean 0 and variance 1, let

$$(2;4) \qquad \Phi(u) \overset{\text{def}}{=} P(u \geq a)$$

and let u_α be defined by $\Phi(u_\alpha) = \alpha$, then (cf. e.g., Noether [7]) the asymptotic power of a test T_n satisfying the above given conditions is $\Phi(u_\alpha - kc)$.

Let C be the class of all tests of H_o satisfying the conditions A-E (or A-E') and suppose C contains a test, T_{no} say, such that for every given α and k no other test in C has a larger asymptotic power than T_{no}; then

$$(2;5) \qquad c_o \geq c \quad \text{for all } T_n \in C.$$

A test T_{no} satisfying (2;5) will be called a best test in C and the following theorem will be proved.

<u>Theorem</u> <u>If $T_n \in C$ with</u>

$$(2;6) \qquad \rho \overset{\text{def}}{=} \lim_{n \to \infty} \rho(t_n, t_{no} | \theta_o) = \lim_{n \to \infty} \rho(t_n, t_{no} | \theta_n)$$

<u>then</u>

$$(2;7) \qquad e(T, T_o) = \rho^2 .$$

<u>Proof.</u>

Consider the tests $T_n(\lambda)$ based on the test statistic

$$(2;8) \qquad t_n(\lambda) = \lambda \frac{t_{no}}{\sigma_{no}(\theta_o)} + (1-\lambda) \frac{t_n}{\sigma_n(\theta_o)} ,$$

where λ is a constant independent of n. It is easily verified that $T_n \in C$ for all λ. Further

$$(2;9) \qquad c(\lambda) = \frac{\lambda c_o + (1-\lambda)c}{\sqrt{\lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)\rho}} .$$

From the fact that $T_n(\lambda) \in C$ for every λ and the fact that T_{no} is a best test in C it follows that

$$c(\lambda) \leq c_o \quad \text{for every } \lambda \tag{2;10}$$

and (2;10) is identical with

$$\lambda^2[c^2-c_o^2-2c_o(c-\rho c_o)] - 2\lambda[c^2-c_o^2-c_o(c-\rho c_o)] + c^2 - c_o^2 \leq 0 \quad \text{for every } \lambda. \tag{2;11}$$

From (2;11) it follows that

$$[c^2-c_o^2-c_o(c-\rho c_o)]^2 - (c^2-c_o^2)[c^2-c_o^2-2c_o(c-\rho c_o)] \leq 0, \tag{2;12}$$

which is identical with $c_o^2(c-\rho c_o)^2 \leq 0$ or

$$\rho = \frac{c}{c_o} = \sqrt{e(T, T_o)} . \tag{2;13}$$

3. Examples

3.1 Tests for the hypothesis that the mean of a symmetric distribution is zero

Let X be a random variable with a continuous symmetric distribution $F(x|\theta)$ with mean θ and let $x_1,\dots,x_n$ be a sample from this distribution. Let $u_1,\dots,u_n$ be the ordered absolute values of the observations; let, for $i=1,\dots,n$, v_i be defined by

$$v_i = \begin{cases} 1 \text{ if } u_i \text{ corresponds to a positive observation,} \\ -1 \text{ if } u_i \text{ corresponds to a negative observation.} \end{cases} \tag{3.1;1}$$

and consider tests for H_o: $\theta=0$ based on test statistics of the form

$$t_n = \sum_{i=1}^{n} a_i v_i , \tag{3.1;2}$$

where, for $i=1,\dots,n$, the weights a_i are given functions of i and n. Examples of test statistics of the form (3.1;2) are e.g.,

1. the sign test with $a_i=1$ for $i=1,\dots,n$,
2. Wilcoxon's signed rank test with $a_i=i$ for $i=1,\dots,n$,

3. van der Waerden's test. In this case the a_i satisfy

$$(3.1;3) \qquad \frac{2}{\sqrt{2\pi}} \int_0^{a_i} e^{-\frac{1}{2}x^2} dx = \frac{i}{n+1} \qquad i=1,\ldots,n.$$

So if $\psi(\alpha)$ is defined by $\alpha = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\psi(\alpha)} e^{-\frac{1}{2}x^2} dx$ then

$$(3.1;4) \qquad a_i = \psi\left(\frac{n+1+i}{2n+2} \right) \qquad i=1,\ldots,n .$$

4. the Fisher-Yates test. For this test a_i is the expected value of the $i^{\underline{th}}$ order statistic of a sample of size n from a χ-distribution. This test is asymptotically identical with the van der Waerden test. (cf. e.g., Lehman [5]).

Another test for H_o that will be considered is the test based on the sample mean

$$(3.1;5) \qquad \bar{x}_n = \frac{1}{n} \sum_{i=1}^{n} x_i = \frac{1}{n} \sum_{i=1}^{n} u_i \ddot{v}_i .$$

The correlation coefficient under H_o of two test statistics of the form (3.1;2) with weights a_i and a_i' respectively is

$$(3.1;6) \qquad \rho(t_n, t_n' | H_o) = \frac{\sum_{i=1}^{n} a_i a_i'}{\sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} a_i'^2}}$$

and the correlation coefficient under H_o between a statistic of the form (3.1;2) and $\bar{x}_n$ is

$$(3.1;7) \qquad \rho(t_n, \bar{x}_n | H_o) = \frac{\sum_{i=1}^{n} a_i Eu_i}{\sigma \sqrt{n \sum_{i=1}^{n} a_i^2}} ,$$

where σ^2 is the variance of X.

Two distributions $F(x|\theta)$ will be considered for X.

I. X has a normal distribution with mean θ and variance 1

Best tests in this case are the test based on the mean, the van der Waerden test and the Fisher-Yates test. So the asmymptotic relative efficiency of the sign test with respect to a best test may e.g. be found from (3.1;6) with $a_i=1$ and $a_i' = \psi(\frac{n+1+i}{2n+2})$ or from (3.1;7) with $a_i=1$, $\sigma=1$ and u_i is the i^{th} order statistic of the absolute values of $x_1,\ldots,x_n$. From (3.1;6) we obtain (cf. also van Eeden and Benard [2])

$$(3.1;8) \qquad e = \rho^2 = \lim_{n\to\infty} \frac{\left(\sum_{i=1}^{n} \psi\left(\frac{n+1+i}{2n+2}\right)\right)^2}{n \sum_{i=1}^{n} \left\{\psi\left(\frac{n+1+i}{2n+2}\right)\right\}^2} = \frac{\left(\int_0^1 \psi\left(\frac{y+1}{2}\right)dy\right)^2}{\int_0^1 \left\{\psi\left(\frac{y+1}{2}\right)\right\}^2 dy} = \frac{2}{\pi}$$

and from (3.1;7)

$$(3.1;9) \qquad e = \rho^2 = \lim_{n\to\infty} \left(\frac{1}{n}\sum_{i=1}^{n} Eu_i\right)^2 = \lim_{n\to\infty} \left(\frac{1}{n}\sum_{i=1}^{n} E|x_i|\right)^2 = (E|X|)^2 = \frac{2}{\pi}.$$

For the Wilcoxon test the asymptotic relative efficiency with respect to a best test may e.g. be found from (3.1;6) with $a_i = i$, $a_i' = \psi(\frac{n+1+i}{2n+2})$

$$(3.1;10) \qquad e = \rho^2 = \lim_{n\to\infty} \frac{\left(\sum_{i=1}^{n} i\,\psi\left(\frac{n+1+i}{2n+2}\right)\right)^2}{\sum_{i=1}^{n} i^2 \sum_{i=1}^{n} \left\{\psi\left(\frac{n+1+i}{2n+2}\right)\right\}^2} = 3\,\frac{\left(\int_0^1 y\,\psi\left(\frac{y+1}{2}\right)dy\right)^2}{\int_0^1 \left\{\psi\left(\frac{y+1}{2}\right)\right\}^2 dy} = \frac{3}{\pi}.$$

The asymptotic relative efficiency of the sign test with respect to the Wilcoxon test follows from (3.1;8) and (3.1;10): $e = 2/3$. This efficiency is not equal to the square of the correlation coefficient between the test statistics. For the correlation coefficient we find from (3.1;6) (cf. also van Eeden and Benard [2])

(3.1;11)
$$\rho = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} i}{\sqrt{n \sum_{i=1}^{n} i^2}} = \tfrac{1}{2}\sqrt{3}\,.$$

II. <u>X has a double exponential distribution with density $\frac{1}{2}e^{-|x-\theta|}$</u>

A best test in this case is the sign test (cf. Ruist [9] or Hoeffding and Rosenblatt [3]). So the asymptotic relative efficiency of the van der Waerden test with respect to a best test follows from the correlation coefficient between the test statistics. This correlation coefficient is independent of the distribution of X, the two tests being simultaneously nonparametric. So the asymptotic relative efficiency of the van der Waerden test with respect to the sign test is $2/\pi$ for a sample from a double exponential distribution, the same as the asymptotic relative efficiency of the sign test with respect to the van der Waerden test for a sample from a normal distribution.

For the Wilcoxon test we find (cf. (3.1;11)) $e = 3/4$ for the asymptotic relative efficiency with respect to a best test.

For the test based on the mean the asymptotic relative efficiency with respect to a best test follows from (3.1;7) with $a_i=1$, σ^2 is the variance of a double exponential distribution with density $\frac{1}{2}e^{-|x-\theta|}$ and u_i is the $i^{\underline{th}}$ order statistic of the absolute value of $x_1,\ldots,x_n$, so

(3.1;12)
$$e = \rho^2 = \lim_{n\to\infty} \left(\frac{\sum_{i=1}^{n} Eu_i}{n\sigma}\right)^2 = \left(\frac{E|x|}{\sigma}\right)^2 = \tfrac{1}{2}.$$

Finally a test analogous to van der Waerden's test will be considered, i.e. we choose the a_i such that (cf. (3.1;3))

(3.1;13)
$$\int_0^{a_i} e^{-x}dx = \frac{i}{n+1} \qquad \text{or} \qquad a_i = \ln \frac{n+1}{n+1-i} \qquad i=1,\ldots,n\ .$$

The asymptotic relative efficiency of this test with respect to a best test follows from (3.1;6) with $a_i=1$ and $a_i' = \ln \frac{n+1}{n+1-i}$

$$(3.1;14) \qquad e = \rho^2 = \lim_{n \to \infty} \frac{\left(\sum_{i=1}^{n} \ln \frac{n+1}{n+1-i}\right)^2}{n \sum_{i=1}^{n} \{\ln \frac{n+1}{n+1-i}\}^2} = \frac{\left(\int_0^1 \ln(1-x)dx\right)^2}{\int_0^1 \{\ln(1-x)\}^2 dx} = \tfrac{1}{2} ,$$

so the test with weights (3.1;13) has the same efficiency as the test based on the mean. The same holds for the normal distribution: the test of van der Waerden has the same efficiency as the test based on the mean.

Remark

As already remarked earlier in this section the van der Waerden test and the Fisher-Yates test are asymptotically identical; their asymptotic relative efficiency is 1 for a sample from any distribution and their correlation coefficient is asymptotically 1. So these tests are, for a sample from e.g. a double exponential distribution, an example of a case where the relation $e = \rho^2$ holds and none of the two tests is a best test.

An example will now be given of this situation with $e \neq 1$. Let $T_{n,1}$ and $T_{n,2}$ be two tests in C with $c_2 \leq c_1$,

$$(3.1;15) \qquad \rho \overset{\text{def}}{=} \lim_{n \to \infty} \rho(t_{n1}, t_{n2} | \theta_o) = \lim_{n \to \infty} \rho(t_{n,1}, t_{n,2} | \theta_n)$$

and $\rho \neq \frac{c_2}{c_1}$. Consider the test based on the statistic

$$(3.1;16) \qquad t_n(\lambda) = \lambda \frac{t_{n1}}{\sigma_{n1}(\theta_o)} + (1-\lambda) \frac{t_{n2}}{\sigma_{n2}(\theta_o)}$$

and choose $\lambda=\lambda_o$ in such a way that the test $T_n(\lambda_o)$ based on $t_n(\lambda_o)$ has maximum efficiency. Then λ_o is the value of λ maximizing $c(\lambda)$ and

$$(3.1;17) \quad \begin{cases} \lambda_o = \dfrac{c_1-\rho c_2}{(1-\rho)(c_1+c_2)} \\ [c(\lambda_o)]^2 = c_1^2 + \dfrac{(c_2-\rho c_1)^2}{1-\rho^2} > c_1^2 . \end{cases}$$

So $T_n(\lambda_o)$ has a higher efficiency than T_{n1} and T_{n2}. Further

$$(3.1;18) \quad \lim_{n\to\infty} \rho^2(t_n(\lambda_o), t_{n1}|\theta_o) = \frac{[\lambda_o + (1-\lambda_o)\rho]^2}{\lambda_o^2 + (1-\lambda_o)^2 + 2\lambda_o(1-\lambda_o)\rho} = \left(\frac{c_1}{c(\lambda_o)}\right)^2$$

$$= e(T_1, T(\lambda_o)).$$

So $T_{n,1}$ and $T_n(\lambda_o)$ are two tests in C for which the relation $e=\rho^2$ holds; that $T_n(\lambda_o)$ is not necessarily a best test follows from the following example; let $x_1,\dots,x_n$ be a sample from a normal distribution with known variance and let $T_{n,1}$ and $T_{n,2}$ respectively be the Wilcoxon test and the sign test. Then $c_1 = \sqrt{\frac{3}{\pi}}$, $c_2 = \sqrt{\frac{2}{\pi}}$ and $\rho = \frac{1}{2}\sqrt{3}$ (cf. (3.1;11)). So $\rho \neq \frac{c_2}{c_1}$ and

$$(3.1;19) \quad [c(\lambda_o)]^2 = \frac{3}{\pi} + 4\left(\sqrt{\frac{2}{\pi}} - \frac{3}{2\sqrt{\pi}}\right)^2 = \frac{20-12\sqrt{2}}{\pi} .$$

A best test in this case has $c_o=1$ and from (3.1;19) it follows that $c(\lambda_o) < 1$, so $T_n(\lambda_o)$ is not a best test.

3.2 Tests for the hypothesis that two distributions are identical

Let X and Y be two independent random variables with distribution functions F(x) and G(y) respectively. Let $x_1,\dots,x_m$ and $y_1,\dots,y_n$ be two samples from F and G respectively, let (with $N = m+n$) $z_1,\dots,z_N$ be the ordered observations in the pooled samples and let

$$(3.2;1) \qquad v_i = \begin{cases} 1 \text{ if } z_i \text{ is an observation of } X \\ 0 \text{ if } z_i \text{ is an observation of } Y \text{ .} \end{cases}$$

The tests to be considered for the hypothesis H_o that F and G are identical have statistics of the form

$$(3.2;2) \qquad t_N = \sum_{i=1}^{N} a_i(v_i - \frac{m}{N}),$$

where, for i=1,...,N, the weights a_i are given functions of i and N. Examples of tests of the form (3.2;2) are

1. Wilcoxon's two sample test with a_i=i
2. van der Waerden's two sample test with $a_i = \psi(\frac{i}{N+1})$,
3. the test of Ansari and Bradley with $a_i = |\frac{i}{N+1} - \frac{1}{2}|$,
4. Mood's test with $a_i = (\frac{i}{N+1} - \frac{1}{2})^2$
5. a test with $a_i = \{\psi(\frac{i}{N+1})\}^2$. In this case the a_i correspond to applying the technique of van der Waerden to Mood's test.

The examples 1 and 2 are both tests for location; 3,4,and 5 are, if it is assumed that X and Y have the same median, tests for scale. Another test for location is the test based on the difference between the sample means

$$(3.2;3) \qquad \bar{x} - \bar{y} = \frac{1}{m}\sum_{i=1}^{m} x_i - \frac{1}{n}\sum_{j=1}^{n} y_j = \frac{N}{mn}\sum_{i=1}^{N} z_i(v_i - \frac{m}{N}) \text{ .}$$

The correlation coefficient under H_o between two statistics of the form (3.2;2) is

$$(3.2;4) \qquad \rho(t_N, t'_N | H_o) = \frac{\sum_{i=1}^{N} a_i a'_i - \frac{1}{N} \sum_{i=1}^{N} a_i \sum_{i=1}^{N} a'_i}{\sqrt{\{ \sum_{i=1}^{N} a_i^2 - \frac{1}{N}(\sum_{i=1}^{N} a_i)^2\}\{ \sum_{i=1}^{N} a_i'^2 - \frac{1}{N}(\sum_{i=1}^{N} a'_i)^2\}}}$$

and the correlation coefficient under H_o between a statistic of the form (3.2;2) and $\bar{x} - \bar{y}$ is

$$(3.2;5) \qquad \rho(t_N, \bar{x}-\bar{y} | H_o) = \frac{\sum_{i=1}^{N} a_i \, Ez_i - \frac{1}{N} \sum_{i=1}^{N} a_i \sum_{i=1}^{N} Ez_i}{\sqrt{N-1}\, \sigma \sqrt{\sum_{i=1}^{N} a_i^2 - \frac{1}{N}(\sum_{i=1}^{N} a_i)^2}} ,$$

where σ^2 is the (common) variance of X and Y.

For the tests for location we will consider the case where X and Y both have a logistic distribution with equal known variance and difference between means θ; for the tests for scale we will consider the case where X and Y both have normal distributions with equal known means and ratio of variance θ.

I. X and Y have a logistic distribution

Let X and Y have distribution functions

$$(3.2;6) \qquad \begin{cases} F(x) = \dfrac{1}{1+e^{-(x-\theta_1)}} \\ G(y) = \dfrac{1}{1+e^{-(y-\theta_2)}} \end{cases}$$

respectively. A best test for testing H_o: $\theta_1 - \theta_2 = 0$ is the Wilcoxon two sample test (cf. Lehman [5]).

So the asymptotic relative efficiency of the van der Waerden test with respect to the Wilcoxon test for samples from logistic distributions is $3/\pi$, the same as the asymptotic relative efficiency of the Wilcoxon test with respect to the van der Waerden test for samples from normal distributions, the two tests being simultaneously nonparametric and the van der Waerden test being a best test in the case of a normal distribution.

For the test based on the difference between the sample means the asymptotic relative efficiency with respect to a best test follows from (3.2;5) with $a_i=i$

$$(3.2;7)\quad e = \rho^2 = \lim_{N\to\infty} \frac{[\sum_{i=1}^{N} i\, Ez_i - EX \sum_{i=1}^{N} i]^2}{(N-1)\,\sigma^2(\sum_{i=1}^{N} i^2 - \frac{1}{N}(\sum_{i=1}^{N} i)^2)} = \frac{36}{\pi^2}\lim_{N\to\infty}(\frac{1}{N^2}\sum_{i=1}^{N} i\, Ez_i)^2,$$

where

$$(3.2;8)\qquad \sum_{i=1}^{N} i\, Ez_i = N\, EX + N(N-1)\, EXF(X) = N(N-1)\, EXF(X) = \tfrac{1}{2}N(N-1).$$

So from (3.2;7) and (3.2;8) we obtain $e = \rho^2 = 9/\pi^2$.

We now consider a test analogous to van der Waerden's test, i.e., we choose a_i such that

$$(3.2;9)\qquad \frac{1}{1+e^{-a_i}} = \frac{i}{N+1} \qquad \text{or} \quad a_i = \ell n\, \frac{i}{N+1-i}.$$

This test has (analogous to the case of the normal and the exponential distribution) the same efficiency as the test based on the difference between the sample means. Its efficiency with respect to a best test follows from (3.2;4) with $a_i=i$ and $a_i' = \ell n\, \frac{i}{N+1-i}$

(3.2;10)

$$e = \rho^2 = \lim_{N\to\infty} \frac{[\sum_{i=1}^{N} i\, \ell n\, \frac{i}{N+1-i} - \frac{1}{N}\sum_{i=1}^{N} i \sum_{i=1}^{N} \ell n\, \frac{i}{N+1-i}]^2}{\{\sum_{i=1}^{N} i^2 - \frac{1}{N}(\sum_{i=1}^{N} i)^2\}\{\sum_{i=1}^{N}\{\ell n\, \frac{i}{N+1-i}\}^2 - \frac{1}{N}\{\sum_{i=1}^{N} \ell n\, \frac{i}{N+1-i}\}^2\}}$$

$$= 12\, \frac{(\int_0^1 x\, \ell n\, \frac{x}{1-x}\, dx)^2}{\int_0^1 (\ell n\, \frac{x}{1-x})^2 dx} = 9/\pi^2 .$$

II. X and Y have a normal distribution

Let X and Y have normal distributions with mean zero and variances θ_1 and θ_2 respectively. Best tests for testing H_o: $\theta_1/\theta_2=1$ are the F-test, based on the ratio of the sample variances and the test of the form (3.2;2) with $a_i = (\psi(\frac{i}{N+1}))^2$ (cf. Klotz [4]).

So for the test of Ansari and Bradley the asymptotic relative efficiency with respect to a best test is found from (3.2;4) with $a_i = (\psi(\frac{i}{N+1}))^2$ and $a_i' = |\frac{i}{N+1} - \frac{1}{2}|$

(3.2;11)

$$e = \rho^2 = \lim_{N \to \infty} \frac{[\sum_{i=1}^{N} |\frac{i}{N+1} - \frac{1}{2}| \{\psi(\frac{i}{N+1})\}^2 - \frac{1}{N}\sum_{i=1}^{N} |\frac{i}{N+1} - \frac{1}{2}| \sum_{i=1}^{N} \{\psi(\frac{i}{N+1})\}^2]^2}{[\sum_{i=1}^{N} \{\psi(\frac{i}{N+1})\}^4 - \frac{1}{N}\{\sum_{i=1}^{N} (\psi(\frac{i}{N+1}))^2\}^2][\sum_{i=1}^{N} (\frac{i}{N+1} - \frac{1}{2})^2 - \frac{1}{N}(\sum_{i=1}^{N} |\frac{i}{N+1} - \frac{1}{2}|)^2]}$$

$$= \frac{[\int_0^1 |x-\frac{1}{2}| \{\psi(x)\}^2 dx - \int_0^1 |x-\frac{1}{2}| dx \int_0^1 \{\psi(x)\}^2 dx]^2}{[\int_0^1 \{\psi(x)\}^4 dx - \{\int_0^1 \{\psi(x)\}^2 dx\}^2][\int_0^1 (x-\frac{1}{2})^2 dx - \{\int_0^1 |x-\frac{1}{2}| dx\}^2]} = \frac{6}{\pi^2} .$$

For Mood's test we find from (3.2;4) with $a_i = \{\psi(\frac{i}{N+1})\}^2$ and $a_i' = (\frac{i}{N+1} - \frac{1}{2})^2$

$e = \rho^2 = \frac{15}{2\pi^2}$.

4. Acknowledgment

The author wants to thank Dr. Charles H. Kraft for the many helpful conversations during the preparation of this paper.

5. References

[1] Cramér, H., Mathematical Methods of Statistics, Princeton University Press, 1946.

[2] Eeden, Constance van and A. Benard, A general class of distribution-free tests for symmetry, containing the tests of Wilcoxon and Fisher, Proc. Kon. Ned. Akad. v. Wet: A60, Indagationes Mathematicae 19 (1957) 381-408.

[3] Hoeffding, W and Joan R. Rosenblatt, The efficiency of tests, Ann. Math. Stat. 26 (1955) 52-63.

[4] Klotz, J., Nonparametric tests for scale, Ann. Math. Stat. 33 (1962) 498-512.

[5] Lehman, E. L., Testing Statistical Hypotheses, J. Wiley and Sons, Inc., 1959.

[6] Noether, G. E., Asymptotic properties of the Wald-Wolfowitz test of randomness, Ann. Math. Stat. 21 (1950) 231-246.

[7] Noether, G. E., On a theorem by Pitman, Ann. Math. Stat. 26 (1955) 64-68.

[8] Pitman, E. J. G., Unpublished lecture notes, Columbia University, 1948.

[9] Ruist, E., Comparison of tests of nonparametric hypotheses, Arkiv för Matematik 3 (1954) 133-163.